ESU-8

Oslo, Norway July 20-24, 2018

TIME MEASUREMENT AS AN INTERDISCIPLINARY SUBJECT IN MATHEMATICS EDUCATION

The Calendar

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THE CONTEXT

Time & the concept of *measurement*

What does it mean: "*we are measuring time*"?

Unlike space, conceiving time lies **beyond** immediate sensory perception

Time concept indissolubly connected with ***time measurement***

- Finding & using **periodic phenomena** ("**clocks**")
- **Time unit(s)** should be "**stable**" (i.e. "time-independent")
- **Clocks** should be **compatible** – **periods' ratios**: **constant**

THE *THREE* BASIC *CLOCKS*

Empirical/Physical fact

- **Earth's rotation** (around its axis): **day-night interchange**

  the **DAY**

- **Earth's revolution** (around the sun): **Seasons' succession**; Fixed stars return to **same** position in the sky

  the **YEAR**

- **Moon's revolution** (around the earth): **Lunar** (four) **phases**

  the **MONTH**; the **WEEK**

THE *SOCIAL* CONTEXT & CONSTRAINTS

- ***Social request*** (political, economic, religious)

Social life should be based on "simple" temporal cycles
i.e. **integral** periods of appropriate periodic phenomena

THE FUNDAMENTAL *MATHEMATICAL* CONSTRAINT

- ***Mathematical fact***

No simple quantitative relation between the periods of the three basic clocks i.e.

- simple **integer relations** (simple rational numbers)
- But, choosing **sufficiently small** time **units rational** relations may result

underlying concept: (real) numbers form a **continuum**

THE *THREE* BASIC *CLOCKS*
quantitative relations – current values

$$t_Y = \frac{\text{year}}{\text{day}} := \frac{\text{mean } \textbf{tropical} \text{ year}}{\text{day } (\textbf{SI} \text{ units})} = \mathbf{365.2421897} = 365^d\, 5^h\, 48'\, 45'',19$$

$$t_M = \frac{\text{month}}{\text{day}} := \frac{\text{lunar mean } \textbf{synodic} \text{ period}}{\text{day } (\textbf{SI} \text{ units})} = \mathbf{29,5305885} = 29^d\, 12^h\, 44'\, 2'',88$$

$$t_D = \frac{\text{day } (\textbf{SI} \text{ units})}{\text{sec}} = \mathbf{86,400}$$

→ **definition** of the **second**

$$t'_D = \frac{\text{day}}{\text{day } (\textbf{SI} \text{ units})} = 86{,}400.003$$

$$\text{(atomic) sec} = \frac{9{,}192{,}631{,}770}{\nu} = 86{,}400.003$$

ν ← ***frequency** of **radiation** due to transition between two hyperfine energy levels of the ground state of* → $^{133}_{55}Cs$

MOTIVATION

The basic questions

1. Why was/is it **important** to determine/specify an accurate/stable **time unit**?

- ***History***: trustful means to reckon historical facts (Thucydides, Varro (AUC), Flavius Josephus, Venerable Bede ....)
- ***Politics***: measure duration of service/term of persons holding public positions (Athenian Archons, Roman Consuls ....)

Regnal years/eras: reference to regularly occurring/important event(s): Olympiads, AUC, AD, Anno Diocletiani, Egira ....)

- ***Economy***: measure/pay for human work, interests of debts (roman *Kalendae*, *Indictions*… human *wages* & rise of *money economy*; request for uniformity of time)
- ***Theology***: **uniform** religious canon (catalytic importance: indisputable determination of day of Christian Easter; Benedictine monasticism…)
- ***Geography/Navigation***: accurate determination of longitude (position at sea; marine chronometer ....)

2. **How** was/is it **possible** to determine/specify an accurate/stable **unit/time-keeping**?

- ***Astronomy***: Study the motion of **celestial** bodies & determine their **periods** accurately (cf. the three basic "clocks")
- ***Physics***: Reveal the **laws** governing **periodic** (mechanical) **phenomena**

Simple pendulum (Galileo): limited to "small" oscillations

Cycloidal pendulum (Huygens): cycloid's properties (tautochrone & involutes) (motivation for accurate clocks)

Modern (chrystal/atomic) clocks: microscopic periodic phenomena

$$SiO_4\,,\ {}^{133}_{55}Cs\,,\ {}^{1}_{1}H$$

- ***Technology***: Construct **devices** operating accurately as **artificial** mutually **compatible periodic** phenomena

  Water clocks

  Sundials

  Mechanical clocks (using springs & pendulums) & the *"escapement mechanism"*

  Modern clocks (based on crystal/atomic vibrations)

INTERRELATIONS

Why was/is it **important**

How was/is it **possible**

to determine/specify an accurate/stable **time units**?

Related with mathematical issues **elementary** and/or **advanced**

by **current standards**!

A short (inexhaustible) list)

Leitmotiv: existence of **positional number system**

- Measuring the **compatibility** of the three "**clocks**" ⟺ • **Fractions** as (periodic) **decimal numbers**

- **Temporal cycles** (regularities among the three *clocks*) ⟺ • **Least common multiple**, • **congruences** (number theory)

- **Ratios** of the three *clocks*' **periods** & search for a convenient **calendar** ⟺ • **continued fractions**

- **data tabulation** (**week day on a given date**; **Easter Sunday** on a given year) ⟺ • (elementary) **algebraic representation** (data parameterization and algebraic symbolism & operations)

Development of **computing algorithms**

- specifying & constructing **accurate** (mechanical) **clocks** (the "**escapement**" mechanism) ⟺ • mathematics of the
  - **Simple** pendulum
  - **Cycloidal** pendulum

  & the **geometry** of (plane) **curves**

Measuring Time as a subject of/in Mathematics Education (*HPM* framework)

I. *Which* history *suitable, pertinent, and relevant for didactical purposes?*

History vs **Heritage**

(mainly) **Heritage** ⟺ *"How did we get there?"*

↓

Calendar & **clock**:

- modern life unthinkable without them
- But **were always there**?

Measuring Time as a subject of/in Mathematics Education (*HPM* framework)

II. *Which role?*

- **Replacement** (Vicariant)
- **Reorientation** (Dépaysant)
- **Cultural** (Culturel)

(mainly) **Cultural** ⟷ **Nowadays** facts & customs taken for granted **emerged** via (simple) mathematics

↓

Strong **influence** of factors – problems - questions

- **social**
- **political**
- **religious**

Measuring Time as a subject of/in Mathematics Education (*HPM* framework)

III. *With which objective?*

- Learning of mathematics
- Development of views on the nature of mathematics & mathematical activity;
- **Teachers' didactical background & pedagogical repertoire**
- Affective predisposition towards mathematics
- **Appreciation of mathematics as a cultural-human endeavour**

Teachers' didactical background ⟷

- **Interrelate** mathematics with other disciplines
- Provide **recreational problems**
- **Enrich** math teaching with **historically important questions** from other domains

Mathematics as a **cultural endeavor** ⟷

- The **calendar** & its emergence
- The **math of the calendar**
- Finding **Easter's date**
- Determining the **geographic longitude**
- Accurate **clocks & navigation**; great geographical expeditions

Measuring Time as a subject of/in Mathematics Education (*HPM* framework)

IV. *In which way?*

History as a ***tool*** vs. History as a ***goal***

(mainly)
History as a ***goal*** ⟺

e.g.
- **Why** it was important & **how** it was possible to **construct** (accurate) **clocks** as a source of **stimulation** of the required **mathematics**

⟺
- The (**religious, political**) **importance** of an accurate **calendar** & the **difficulties without** (nowadays) **"simple" mathematics**:
  - **algebraic symbolism**
  - elementary concepts in **number theory**
  - .....

Measuring Time as a subject of/in Mathematics Education (*HPM* framework)

V. *Following which approach?*

- Providing **direct historical information**
- Implementing a teaching approach inspired by history (explicitly or implicitly)
- Focusing on **mathematics** as a **discipline** and the **cultural & social context**

(mainly) direct historical information

(mainly) mathematics as a discipline and the cultural & social context

⟺

e.g.

Looking for the "**optimal**" ***Leap Year rule***

Omar Khayyam **8/33** rule
vs
Gregorian **97/400** rule

Measuring Time as a subject of/in Mathematics Education (*HPM* framework)

VI. *which methodological scheme?*

- **Illumination** approach
- **Module** approach
- History-based approach

e.g.

Illumination approach ⟷ ***The week day of a given date***:
- **various methods** and their **history**
- relation to **congruences** in number theory
- the (origin of) astronomers' ***Julian date***

Module approach ⟷
- The **calendar**: Its **history & mathematics**
- **Measuring time**: The **clock**, its history & the underlying **physico-mathematical theories**

EXAMPLES

1. The astronomers' *Julian date*:

Its origin, Gauss & the *Chinese remainder theorem* in Number Theory

2. The optimal "*leap year rule*" & *continued fractions*:

Julius Caesar, Omar Khayyam & Pope Gregory XIII

3. The week day on a given date:

From old "*dominical letters*" to Gauss, Lewis Caroll & modern computer algorithms ("*odd-plus 11- method*)

4. *Computus* & *Easter* Sunday - The struggle for an accurate solar calendar:

A meeting point of *Ancient Greek Astronomy*, *time reckoning in the Roman Empire*, *Jewish lunar calendar & festivities*

5. Accurate clocks, the *escapement mechanism* & the underlying physico-mathematical theories:

From Galileo & Huygens, to atomic vibrations

TEMPORAL CYCLES

A mixture of astronomical facts & social conventions

- **Metonic cycle** (Meton, Athens 432BC): **19 years**

Tropical **year*** — $t_Y = 365^d.2422$

Synodic **month** (**lunation**): — $t_M = 29^d.53059$

l.c.m. (t_Y, t_M) $\mathbf{19}t_Y \cong \mathbf{235}t_M \cong \mathbf{6940^d}$

Lunar phases **repeated** on the **same** **date** **every 19 years** (approximately)

*(Julian) year $=365^d.25 = (365+ 1/4)^d$

- **Callipic cycle** (Callipus, Cyzicus ~330BC): **76 years** (4×19)

$$\underbrace{\frac{6940}{19}}_{\text{Metonic cycle}} = 365 + \frac{5}{19} = \underbrace{365 + \frac{1}{4}}_{\text{Julian year}} + \underbrace{\frac{1}{4 \times 19}}_{\text{Callipic cycle}}$$

Callipic cycle: 76 Julian years with 1 day omitted at the cycle's end

Comment: Months per year & continued fractions

$$\frac{t_Y}{t_M} = 12.36827 = 12 + 0.36827$$

$$0.36827 = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{17 + \cfrac{1}{287/218}}}}}}} =$$

Convergents of 0.36827	1/2	1/3	3/8	4/11	7/19	123/334
t_Y/t_M	25/2	37/3	**99/8**	136/11	**235/19**	4131/334
Decimal approximation	12.5	12.333	12.375	12.364	12.3684	12.3682
			Octaeteris		**Metonic** cycle	

- **Solar cycle**: **28 years** (4×7)

(related to the ***Julian year*** & the ***week***)

Every **28** years **each** date of the year falls on the same **same week day**

Because (l.c.m.(7,4)=28)

$$365 = 52\times7 + 1 \equiv 1 \bmod 7 \quad \& \quad 1 \text{ leap year every 4 years}$$

Remark: By **convention** start on a **leap year** with **1/1 Monday**

- **Indictions**: **15 years** (*Indictio*) Fiscal period:

- Roman Egypt (3rd century AD): 5-year periodic reassessment of agricultural/land taxes
- Constantine, Constantinople 312AD: 15-year starting from 313AD

Hence for year **Y** (AD)

$$\text{Indiction Year} \equiv (\mathbf{Y}+2)\ \mathbf{mod}15 + 1$$

Remark: Dating in Byzantine empire, medieval Europe, Holy Roman Empire (till Napoleon)

EXAMPLES

1. The astronomers' *Julian date*:

(J.J. Scaliger 1582; starting on 1/1/4713BC!? – measured in days)

Its origin, Gauss, & the *Chinese remainder theorem* in Number Theory

- History: Dionysius Exiguus (525AD) establishes

AD era (***Anno Domini***): **Start** time-reckoning on **1AD**

- Astronomical fact: New moon 23/3/323 ⇒ 1/1/325

Council of Nicaea

- Nicene Creed
- Easter celebration canon
- Sunday – day of rest

Start new Metonic cycle (1^{st} year) & **323 ≡ 0 mod19** imply 323AD & 1BC are 1^{st} years

1AD: **2^{nd}** year of **Metonic** cycle

- **Indiction** cycle

For year **Y**(AD) Indiction Year ≡ (**Y**+2) **mod**15 +1, hence

1AD: **4th** year of **Indiction** cycle

- **Solar** cycle

- Known: 1/1/ 325AD was a Friday, hence
- 328AD (leap year with 1/1 Monday), i.e. Start of new solar cycle

so, for year **Y** (AD): Solar cycle Year ≡ (**Y**+9) **mod**28, hence

1AD: **10th** year of **Solar** cycle

- **J.J. Scaliger** (1583): *Opus novum de emendatione temporum* (***Computi*** *Annales* - calendrical calculations)

- To disentangle chronology from absoluteness of religious creeds & unreliable records, start counting by sufficiently going backwards in time

- To avoid difficulties of reconciling the 3 “clocks”

- **Mathematical problem**

(Gauss *Disquisitiones Arithmeticae*, 1801)

Look for the year (beginning of a **new** era), such that for x=1AD

$$x \equiv \mathbf{10} \bmod \mathbf{28}, \quad x \equiv \mathbf{2} \bmod \mathbf{19}, \quad x \equiv \mathbf{4} \bmod \mathbf{15}$$

Since (28, 19, 15) are pairwise relatively prime ⇒ solution exists

Chinese Remainder Theorem:

$x \equiv (10x_1 + 2x_2 + 4x_3) \bmod \mathbf{7980}$, $\quad 7980 = 28 \times 19 \times 15$

$x_1 = 19 \times 15\, y_1 \equiv 1 \bmod \mathbf{28}$

$x_2 = 19 \times 15\, y_2 \equiv 1 \bmod \mathbf{28}$

$x_3 = 19 \times 15\, y_3 \equiv 1 \bmod \mathbf{28}$

$$\boldsymbol{x = 4714}$$

Julian cycle (converted to days): **7980** years

Start of Julian cycle/era: **4713**BC

Remarks: (a) Julian date used by astronomers (Julian **day** number – JDN)

(b) Develop algorithms to calculate JDN for any date!

EXAMPLES

2. The optimal "*leap year rule*" & *continued fractions*: Julius Ceasar, Omar Khayyam & Pope Gregory XIII

Serious difficulties:

I. No unique definition of the three basic **"clocks"** because of

- the relative motion(s): sun – moon – earth
- Interactions among: earth – sun – moon - planets

Year: sidereal, **(mean) tropical**, anomalistic, lunar

Month: sidereal, **(mean) synodic**, anomalistic

Day: sidereal , apparent solar, mean solar, **day (SI)**

II. Both day & the (mean) **tropical year** are important

III. Tropical **year/day** is **not integer**

0th approximation (Roman calendar): $365^d = 12 \times 30^d + 5^d$ (intercalation)

1st approximation (Julius Caesar, 46BC): $365^d.25$

4-year cycles: 3×365^d (common year) $+1 \times 366^d$ (leap year)

IV. Tiny measurement errors due to limited accuracy of observations; **observable** effects over **long** time intervals (**centuries**) – important for reasons

Religious: regular & strict celebration of festivities (e.g. Christian Easter)

Historical: reckoning facts in different epochs at different places

Political: seasonal activities related to society, economy, agriculture

t_Y year current value	t_{JY} Julian year	t_M month (mean synodic)	t'_Y Alfonsine tables (1252/1518)	t_D day current value
$365^d.2422$	$\mathbf{365^d.25}$	$29^d.53059$	$\mathbf{365^d.242546}$	86,400 sec(SI)
$365^d 5^h 48' 46''$	$365^d 6^h$	$29^d 12^h 44' 2''$	$365^d 5^h 49' 16''$	

$t_{JY} - t'_Y = 10' 4'' =$ **0.0074537** days/year, or **1 day** lost every **134.16 years**

Proposed corrections:

P. d' Ailly (1412): Omit 1 leap year (1 day) every 134 years

P. Pitatus (1568): Omit 3 leap years (3 days) every 400 years:

$$\frac{3}{400} = \frac{1}{133 + \frac{1}{3}} \cong \frac{1}{134}$$

A. Lilius, C. Clavius (Gregorian reform, 1582): Same rule with omitted leap years are centurial years not divisible by 4

Remark: (a) Alfonsine tables: day's fractions written in sexagesimal system

$$365^{\mathrm{d}}\,5^{\mathrm{h}}\,49'\,16'' = 365^{\mathrm{d}} \left(\mathbf{\frac{14}{60}}\right)\left(\mathbf{\frac{33}{60^2}}\right)\left(\frac{9}{60^3}\right)\left(\frac{57}{60^4}\right)$$

$$\left(\frac{14}{60}\right)\left(\frac{33}{60^2}\right) = \frac{873}{3600} = \mathbf{\frac{97}{400}}$$

(b) Lagrange (1774): $365^{\mathrm{d}}5^{\mathrm{h}}49'20''=365^{\mathrm{d}}\,20960''$ & **20952/86400** = **97/400**

Euler: Tropical year ($365^{\mathrm{d}}\,5^{\mathrm{h}}48'55''$) as a **continued fraction** to justify Gregorian leap-year-rule.

Questions:

- Since there is **always** an **error** left, **is there any better leap-year-rule**?
- Can the **two rules** (Julian, Gregorian) be described in a **unified** way?

Continued fractions: Introduced by independent authors in various contexts: Bombelli, Huygens, Wallis, Euler ….

$$a = [a] + \frac{1}{\frac{1}{a-[a]}} = [a] + \frac{1}{\frac{1}{a'-[a']}} = \cdots \text{etc}, \quad a' = \frac{1}{a-[a]}$$

Recursive theorem (Wallis, Euler): convergents A_n/B_n of a

$$a = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{..+\cfrac{1}{..+\cfrac{1}{b_n}}}}}} = \frac{A_n}{B_n}$$

$$A_n = b_n A_{n-1} + A_{n-2}\ , \qquad B_n = b_n B_{n-1} + B_{n-2}$$

$$A_{-1} = 1, \qquad A_0 = b_0, \qquad A_1 = b_1 b_0 + 1\ ,$$
$$B_{-1} = 0, B_0 = 1, B_1 = b_1$$

Continued fraction *expansion of the tropical year* t_Y =365^d.2422

$$0.2422 = \frac{2422}{10000} = \frac{1211}{5000} = \cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}}}}}$$

Convergents of 0.2422	$\frac{1}{4}$	$\frac{7}{29}$	$\frac{8}{33}$	$\frac{31}{128}$	$\frac{132}{545}$	$\frac{163}{673}$	$\frac{295}{1218}$	$\frac{458}{1891}$	$\frac{1211}{5000}$

Julian calendar (46BC): **¼=0.25**

Omar Khayyam (c.1076AD): **8/33** = 0.2424.... **better**(!) than

Gregorian calendar (1582AD): **97/400** = 0.2425

Other possibilities(!): **31/128**: 128 = 32 × 4 -1 = 0.24219

1211/5000: 1211=1250 - 50+ 10 + 1 =0.2422

EXAMPLES

3. The week day on a given date:

From old “*dominical letters*” to Gauss, Lewis Caroll & modern computer algorithms (“*odd-plus 11- method*”)

solar cycle

Temporal cycles involved: year , week , month

astronomical semi-astronomical/civil

Elementary mathematics involved, but **no simple rule** exists, because:

(i)

365≡1mod7	Each date moves	1	day(s) forward per	common	year
366≡2mod7		2		leap	

(ii) 1 leap year every 4 year + Gregorian correction for centurial years

(iii) Irregular distribution of days among months for historical/incidental reasons (**4** with 30, **7** with 31, **1** with 28/29)

Hence, subject **fascinating** for reasons:

political, religious, recreational, mathematical

It can be solved in different, progressively formalized ways:

(i) Four ***arithmetical operations*** **&** ***data tabulation*** - extensive use of 28-year solar cycle; inconvenient for Gregorian calendar

(ii) Elementary use of ***congruences*** (Number Theory) & ***algebraic symbolism*** to parameterize dates & week days, and make **minimal** use of ***data tabulation***

(iii) More ***elaborate formalization*** (similar to (ii)), capable for ***computer implementation***

Remarks: Several **different equivalent** formulations exist:

(a) Mathematically interesting/insightful to prove/check equivalence

(b) All use some numbering of week days (mapping to a arithmetical set). Here:

Sun	Mon	Tue	Wed	Thu	Fri	Sat	**Weekday number**
1	2	3	4	5	6	7	

Examples

Parametrization:

Year AD		Month	Day of month	date	Weekday
Y	**Y=100c+y**	**m**	**d**	**d/m/Y**	**W**

I. Gauss (unpublished):

For 1/1/Y: W=2+(5Y **mod4** + 4Y **mod100** + 6Y **mod400**) **mod7**

For d/m/Y (variation) with:

Mar/Y: m=1, … Dec/Y: m=10, (Jan/Y, Feb/Y): m=(11, 12) for Y-1

$$W = 1 + (d + [(13m-1)/5] + y + [y/4] + [c/4] - 2c) \bmod 7$$

II. Zeller (1883), with m′=m+2

$$W = (d + [13(m'+1)/5] + y + [y/4] + [c/4] - 2c) \bmod 7$$

III. A. De Morgan (1845?), (slightly adapted)

- **I.** Add 1 to the given year.
- **II.** Take the quotient found by dividing the given year by 4 (neglecting the remainder).
- **III.** Take 16 from the centurial figures of the given year if that can be done.
- **IV.** Take the quotient of III divided by 4 (neglecting the remainder).
- **V.** From the sum of I, II and IV, subtract III.
- **VI.** Find the remainder of V divided by 7, and subtract from 7: this is the weekday number of 1/1.

$$W = 7 - \left(1 + Y + [Y/4] + [(Y-1600)/400] - [(Y-1600)/100]\right) \bmod 7$$

IV. Lewis Carroll (1887):

Take the **given date** in **4 portions**, viz. the number of centuries, the number of years over, the month, the day of the month. **Compute the following 4 items**, adding each, when found, to the total of the previous items. When an item or total exceeds 7, divide by 7, and keep the remainder only.

- **Century-item**: For '*Old Style*' (which ended 2 September 1752) subtract from 18. For '*New Style*' (which began 14 September 1752) divide by 4, take overplus from 3, multiply remainder by 2.
- **Year-item**: Add together the number of dozens, the overplus, and the number of 4s in the overplus.
- **Month-item**: If it begins or ends with a vowel, subtract the number, denoting its place in the year, from 10. This, plus its number of days, gives the item for the following month. The item for January is "0"; for February or March, "3"; for December, "12".
- **Day-item**: The total, thus reached, must be corrected, by deducting "1" (first adding 7, if the total be "0"), if the date be January or February in a leap year, remembering that every year, divisible by 4, is a Leap Year, excepting only the century-years, in 'New Style', when the number of centuries is not so divisible (e.g. 1800).

The final result gives the day of the week, "0" meaning Sunday, "1" Monday, and so on.

v. The odd-plus-11 method (Chamberlain & Walters 2010)

<u>For 1/1/Y</u>:

Y=100*c*+y,

W_{00} = weekday number for 1/1 of 100c (tabulated)

$$W=7-\Big[\big((y+11y)/2 \text{ mod}2\big)+11\big(\big(y+11y \text{ mod}2\big)/2\big)\text{mod}2\big)\Big]\text{mod}7-W_{00}$$

A *hybrid* method – semi-formalized, semi-tabular

Calendrical number *of a given date in a year* (from 1/1 to 31/12):

- canonical mapping: $\{1, 2, 3, \ldots, 365\} \rightarrow \mathbb{Z}_7$

(important) ***Convention***: 29/2 **&** 1/3 → 60 ≡ 4 **mod7**

- identify: {1, 2, 3, 4, 5, 6, 0} ≡ {A, B, C, D, E, F, G}

calendrical

number letter

Remarks:

(a) **Each weekday** in a (common) year has the **same** calendrical **letter**, throughout the year (same image in $\mathbb{Z}_7$)

(b) Define ***Dominical letter/number*** (of the year Y)

$\mathbf{N_Y}$= calendrical letter/number of **Sundays**

Determines the **weekday of 1/1/Y** (14 alternatives; common or leap year)

		Dominical Number/Letter			
Week day, 1/1	W	Common Year (N)		Leap Year (N\N-1)	
Sunday	1	1	A	1\7	A\G
Saturday	7	2	B	2\1	B\A
Friday	6	3	C	3\2	C\B
Thursday	5	4	D	4\3	D\C
Wednesday	4	5	E	5\4	E\D
Tuesday	3	6	F	6\5	F\E
Monday	2	7	G	7\6	G\F
		For the **whole** year		Jan, Feb \ Mar-Dec	

Forward in time ↑

$$W_{1/1/Y} + N_Y \equiv 2 \bmod 7$$

Define ***Regular of month*** **m**, $\mathbf{R_m}$ (**independent** of the year)

$$\mathbf{R_m} = (\text{number of } 1/m)\ \mathbf{mod}7$$

m	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
$\mathbf{R_m}$	1	4	4	0	2	5	0	3	6	1	4	6

- Weekday number: $W_{d/m} \equiv (W_{1/1} - 1 + R_m + d - 1 + \delta)\ \mathbf{mod}7$

$$\delta = \begin{cases} 0\ ;\ (\text{Y common year \& } m \leq 2) \\ 1\ ;\ (\text{Y leap year \& } m \geq 3) \end{cases}$$

Remark: The problem reduces to determining $\mathbf{W_{1/1/Y}}$ uniquely specified by the dominical number $\mathbf{N_Y}$:

common year: $\mathbf{N_{Y+1} = N_Y - 1}$, $\mathbf{W_{1/1/Y+1} \equiv (W_{1/1/Y} + 1)mod7}$

leap year: $\mathbf{N_{Y+1} = N_Y - 2}$, $\mathbf{W_{1/1/Y+1} \equiv (W_{1/1/Y} + 2)mod7}$

$$W_{1/1/Y} \equiv (W_{1/1/1} + Y\text{-}1 + [Y/4] - [Y/100] + [Y/400]) \ \mathbf{mod7}$$

Hence: sufficient to know W for some reference date

$$W_{d/m/Y} \equiv (d - 1 + R_m + Y\text{-}1 + [Y/4] - [Y/100] + [Y/400] + a) \ \mathbf{mod7}$$

e.g. 1/1/2018 was Monday $\Rightarrow W_{1/1/2018} = 2$

$$W_{1/1/2018} \equiv (2507 + a) \ \mathbf{mod7} \equiv (1 + a) \ \mathbf{mod7} = 2 \Rightarrow a = 1$$

Therefore

$$\mathbf{W_{d/m/Y}} \equiv (d + R_m + Y\text{-}1 + [Y/4] - [Y/100] + [Y/400] - \delta) \ \mathbf{mod7}$$

$$\delta = \begin{cases} 0; & (Y \text{ common year}) \text{ or } (Y \text{ leap year } \& \ m \geq 3) \\ 1; & (Y \text{ leap year } \& \ m \leq 2) \end{cases}$$

EXAMPLES

4. *Computus* & *Easter* Sunday - The struggle for an accurate solar calendar:

A meeting point of *Ancient Greek Astronomy*, *time reckoning in the Roman Empire*, *Jewish lunar calendar & festivities*

Extremely ***rich*** *subject*:

Interrelated aspects

- Historically fascinating
- Intercultural
- Interdisciplinary
- **Mathematically** elementary, but non-trivial

four arithmetic operations

positional number system:
fractions ↔ decimals

algebraic symbolism -
modelization

Complications because

Use of integers only

Astronomical/physical
phenomena

civil & historical
reasons

Historical facts

- **Easter** celebration most important for Christianity
- **Date** originally (up to 3rd century AD) **dependent** on Jewish ***Passover***
- **Council of Nicaea** (325 AD) – request to celebrate Easter:
  - on the same date everywhere;
  - non-coincidence with Jewish *Passover*;
  - Alexandria's church in charge

Basic convention: Easter celebration on

1st Sunday following → week

1st full moon after → lunar synodic period

March 21 (ecclesiastical *vernal equinox*) → (tropical) year

intermingled

- ***Lunar phases*** on same date (d/m/Y): ***Metonic cycle*** (19 years)
- Alexandrians used *Metonic cycle*
- **Calendar** repeated: ***Solar cycle*** (7×4=28 years)

Easter (Paschal) **cycle**: **Easter Sunday dates** repeated

$$28 \times 19 = 19 \times 4 \times 7 = 76 \times 7 = \mathbf{532} \text{ years}$$

Solar (28) → Astronomical & civil convention

Metonic (19) → Astronomical

Callipic (76)

Victorian (532) (Victorius of Aquitaine, 5th AD) → Astronomical & civil convention

- ***Dionysius** Exiguus* (525 AD):

- introduced AD era
- implemented Metonic cycle starting from 1BC

- *Venerable **Bede*** (725AD) – "*De temporum ratione*" (influential in ***Computus***)

- Easter dates till 1064AD
- simple counting & extensive use of tables

Complications

1.

Lunar synodic period:
- not integral number of days: $t_M/t_D \neq$ integer
- < civil month(s)

2. **Wrong** procedure over **long** **periods**, because

Use of ***Julian year*** ($365^d.25$) instead of tropical ($t_Y = 365^d.2422$)

Use of ***notional lunation*** ($6940/235 = 29^d.5319$) instead of $t_M = 29^d.5306$

Thus

- Vernal equinox
- Notional lunations starting

progressively **earlier** than:
- 21/3
- true new moons

t_Y current value	t_{JY} Julian year	t_M (lunar synodic period)
$365^d.2422$	$365^d.25$	$29^d < 29^d.5306 < 30^d$
$19t_Y = 6939^d,60$	$19t_{JY} = 6939^d,75$	$235t_M = 6939^d,69$
rounded to **6940^d** (**Metonic** cycle)		
	physical fact	$29^d < t_M < 30^d$
		$12t_M < t_{JY} < 13t_M$
never used original (roman) final (current)	Civil year: $365^d = 12$ months $= 7 \times 30 + 5 \times 31$ $= 5 \times 30 + 6 \times 31 + 1 \times 29$ $= 4 \times 30 + 7 \times 31 + 1 \times 28$	$t_{JY} = 29^d.5 \times 12 + 11^d.25$ $= 354^d + 11^d.25$

Metonic cycle = **228** (19 × 12) **civil months** = **235 notional lunations**
= **6940^d**

Consider **integer numbers** of **lunations**, i.e. 29^d & 30^d:

$$\left.\begin{array}{l} 29x + 30y = 6940 \\ x + y = 235 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = 110 \\ y = 115 \end{array}\right.$$

No convenient division of civil year to 12 months!

Adopted solution (mathematically **not elegant!**)

- Metonic cycle = 235 notional lunations ("months" **m**) :

235m = **12**y × **12**m/y + **7**y × **13**m/y =

= **12**y × **6**m/y × (30d/m + 29d/m) +

+ **6**y × [**7**m/y × 30d/m + **6**m/y × 29d/m] +

+ **1**y × [**6**m/y × 30d/m + **7**m/y × 29d/m] + **5**d

- Paschal full moon = 14th day of notional lunation **on** or **after** 21/3
  earliest: on 21/3; latest: on 18/4 (full moon on 20/3 + 29d)
- Easter Sunday **S**: earliest on 22/3; latest on 25/4 (full moon on Sunday 18/4)
- Golden number $\mathbf{G_Y}$ of year (AD) **Y**: position of **Y** in Metonic cycle

$$\mathbf{G_Y} \equiv 1+\mathbf{Y}\,\mathrm{mod}\,19$$

- {21/3, 22/3 ... 25/4} ⟶ R = {21, 22, ..., 56}
- Next day of Paschal full moon: $\boldsymbol{r}_\mathbf{Y}$

$\mathbf{S}, \boldsymbol{r}_\mathbf{Y} \in R$

Calculation of Easter Sunday $\mathbf{S_Y}$ of year (AD) **Y**

$$\mathbf{S_Y} \equiv \boldsymbol{r}_\mathbf{Y} + (7 + \mathbf{N_Y} - \mathbf{C})\ \mathbf{mod}\ 7$$

Where

- $\mathbf{C} \equiv (\boldsymbol{r}_\mathbf{Y} + 3)\ \mathbf{mod} 7$, *calendrical number* of $\boldsymbol{r}_\mathbf{Y}$
- $\boldsymbol{r}_\mathbf{Y} \equiv (75 - E_Y)\ \mathbf{mod} 30$
- $\mathbf{E_Y} \equiv (11\ \mathbf{G_Y} - 3)\ \mathbf{mod} 30$,
  $\mathbf{E_Y}$: the *epact* of **Y**= age of notional moon on 1/1/Y
- $\mathbf{G_Y} \equiv 1 + \mathbf{Y}\mathbf{mod} 19$, *golden number* of **Y**
- $\mathbf{N_Y} \equiv (9 - W_{1/1/Y})\ \mathbf{mod} 7$, *dominical letter* of **Y**

EXAMPLES

5. Accurate clocks, the *escapement mechanism* & the underlying physico-mathematical theories:

From Galileo & Huygens, to atomic vibrations

Need of **accurate time-keeping** (measuring **ever smaller time intervals**)

Devices based on { continuous / **oscillatory** } processes { (e.g. water-clocks) / (e.g. mechanical clocks) }

Shift due to use of the ***verge escapement*** mechanism

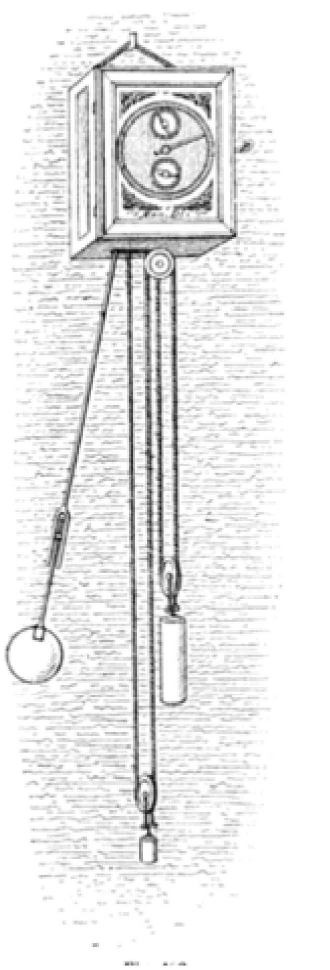
Inaccurate due to swinging across **wide** arcs (~100°)

Two important relevant cases (**technology** & **math** interconnected):

- ***Simple pendulum clock***: Galileo c.1602, 1641 & Viviani 1659; Huygens & Closter 1656-58
- ***Cycloidal pendulum***: Huygens 1673 (*Horologium Oscillatorium*)

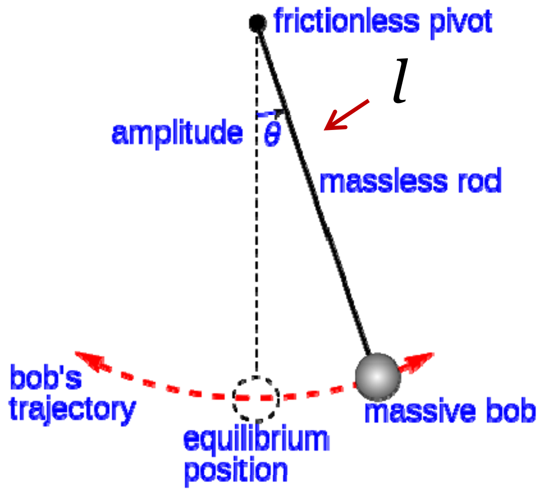
The simple pendulum

Galileo's discovery of (**approximate**) isochronism (c.1602)

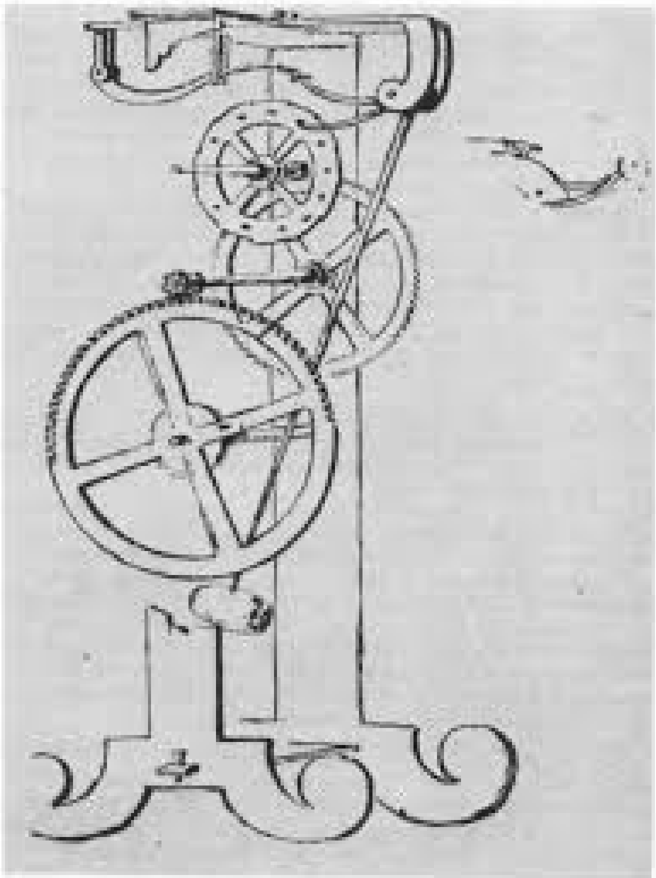


$$\theta''(t) = \frac{g}{l}\sin\theta \approx \frac{g}{l}\theta,\ \ \theta \ll 1 \implies T = 2\pi\sqrt{\frac{l}{g}}$$

accuracy limited to **small** oscillation **amplitudes**



Galileo's pendulum clock (1641, 1659)



Cycloidal pendulum*, *tautochrone*, *involutes

Huygens sought & conceived (more) **accurate clocks**

- to study the **cycloid**
- to introduce **geometrical** concepts (***involutes*** of a curve)
- to prove important properties of the cycloid:
  - Property of ***tautochrone***
  - ***Involutes*** of a cycloid are ***identical cycloids***
- to construct (in principle more) ***accurate clocks***

The **cycloid** & the ***tautochrone***

point mass moving along a cycloid under its weight

v: tangential speed

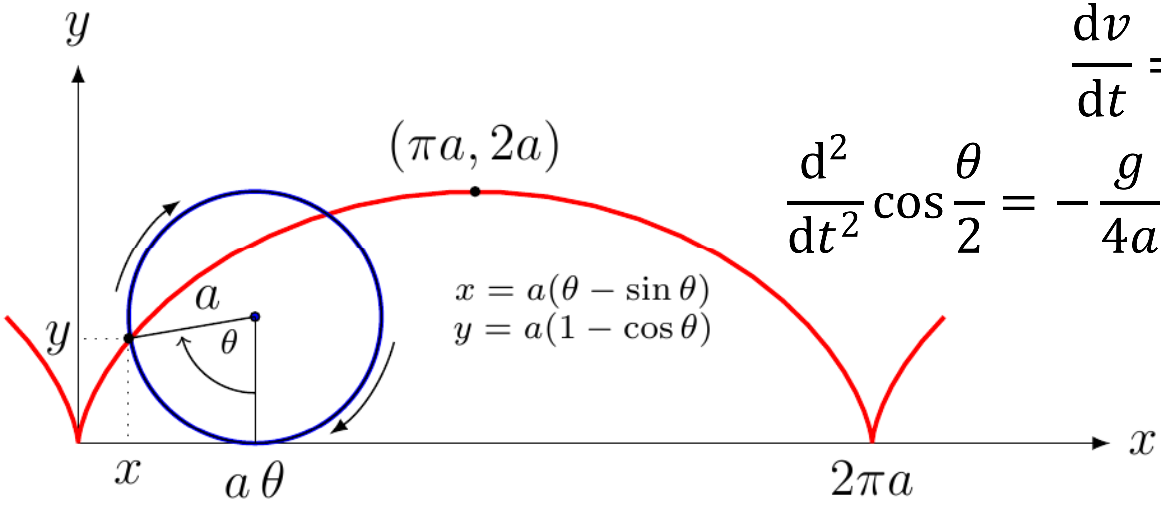
s: arc length

g: gravitational acceleration

y: vertical coordinate

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g\frac{\mathrm{d}y}{\mathrm{d}s}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos\frac{\theta}{2} = -\frac{g}{4a}\cos\frac{\theta}{2}, \textbf{linear} \implies T = 4\pi\sqrt{\frac{a}{g}}$$



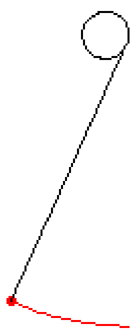
Isochronous
irrespective of initial point

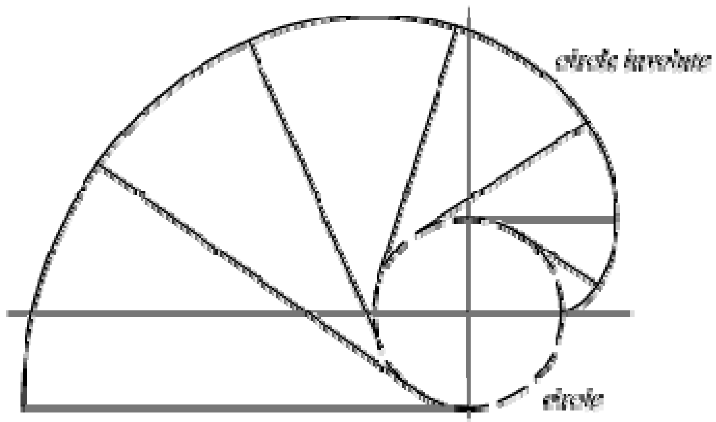
Geometry of curves: ***involutes*** & ***cycloids***

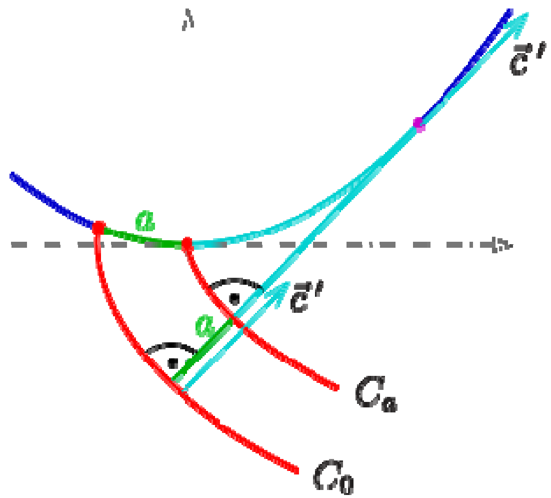
curve c_I	curve c_E
Locus of ***free end*** of a taut ***string*** attached to a point of Curve c_E as the string is ***wound*** along c_E	**Locus** of ***center of curvature*** of curve c_I
Involute of c_E	**Evolute** of c_I

c_I orthogonal to tangents to c_E

spiral	circle



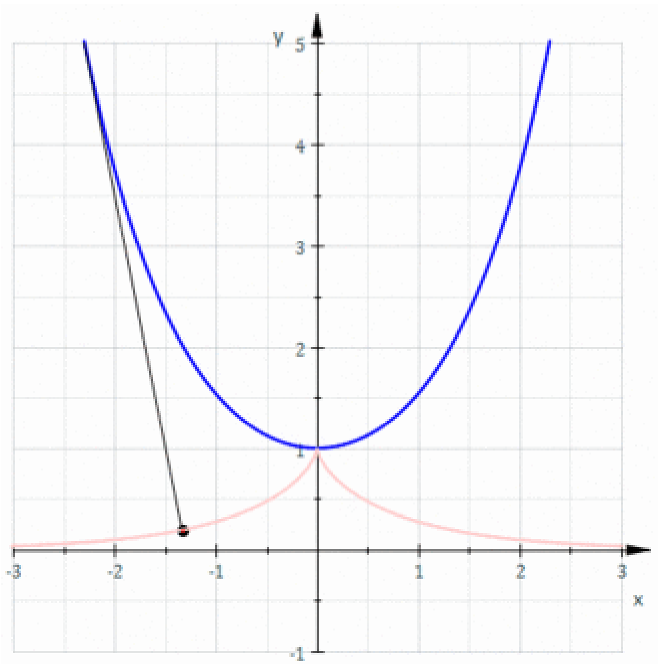


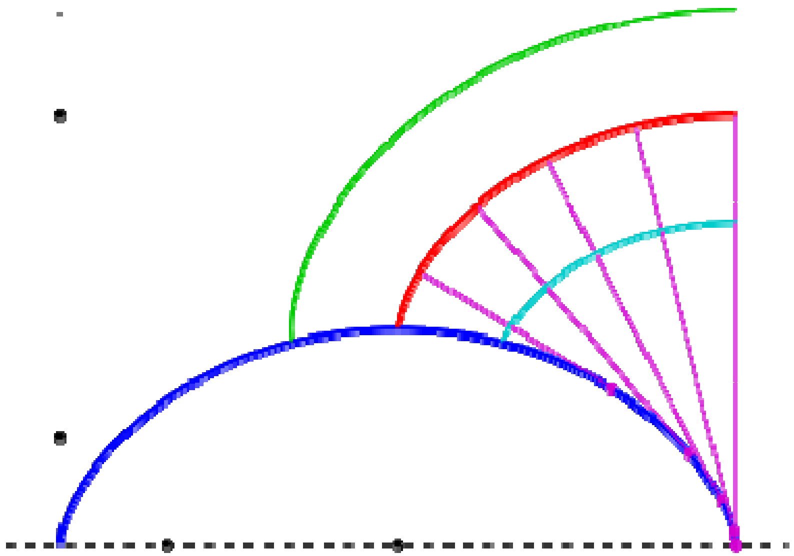


curve c_I	curve c_E
Involute of c_E	**Evolute** of c_I

c_I orthogonal to tangents to c_E

tractrix	catenary
cycloid	**cycloid**





Cycloidal pendulum

Tautochrone & *"self-involuteness"* **combined**!

Point mass pendulum

(***constrained*** ***by a cycloid***)

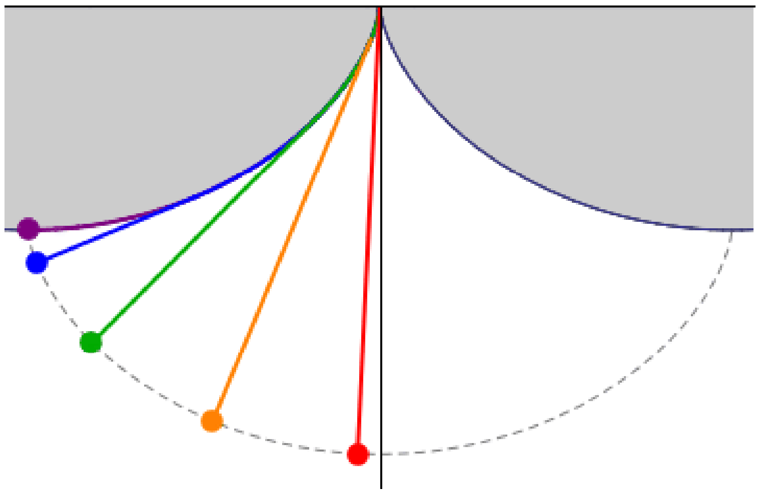
↓

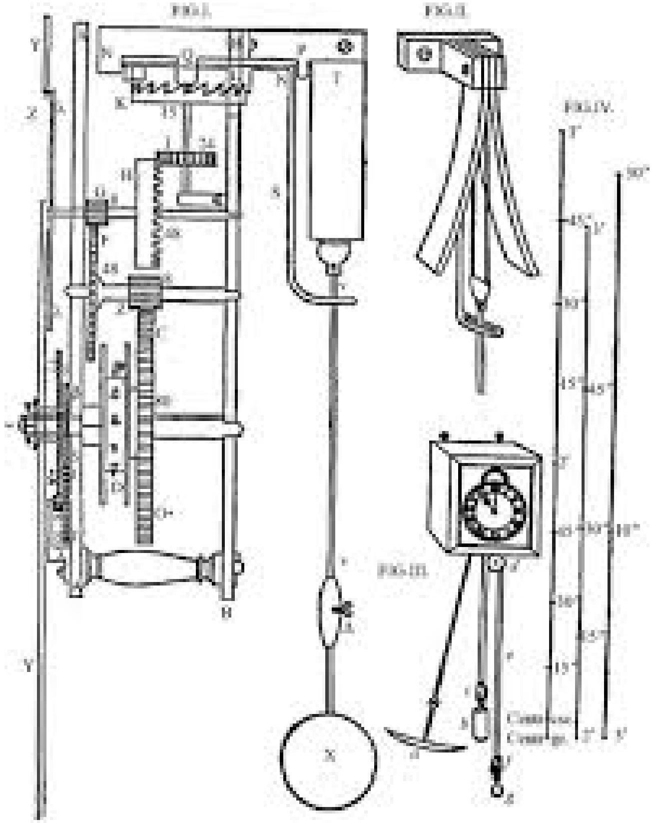
moves along a **cycloid**

↓

period is **amplitude-independent**

(tautochronous)





Mechanism of clocks

A clock consists of **three** parts

I. *Energy source* e.g.

water →	Water clocks
Weight → Springs (elastic potential energy)	Mechanical clocks
electricity →	Quartz clocks

II. *Regulator* (time-keeping element - oscillator; to beat out the "ticks") e.g.

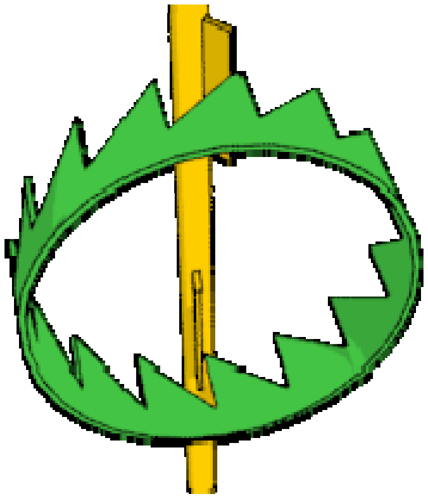
	periodic filling of scoops with water	water clocks
Possess natural frequency (**resist vibration** at **other** frequencies)	balance wheel (foliot) pendulum	mechanical clocks
	quartz crystal	quartz clocks
	vibrating atom	atomic clock

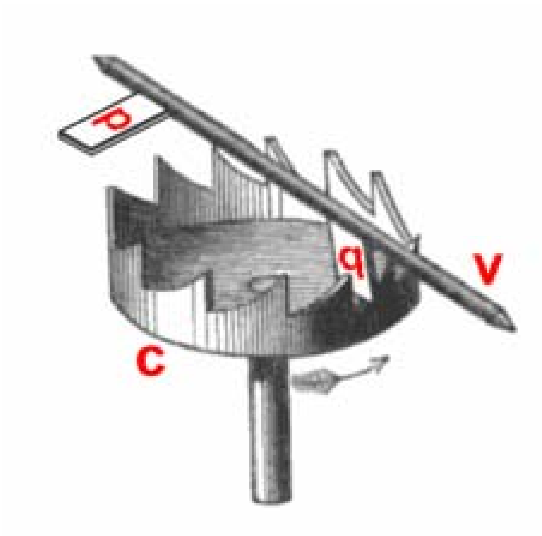
III. ***Escapement mechanism*** (control energy release):

- "push" oscillator to replace energy loss
- Convert oscillator's vibrations into pulses measuring time

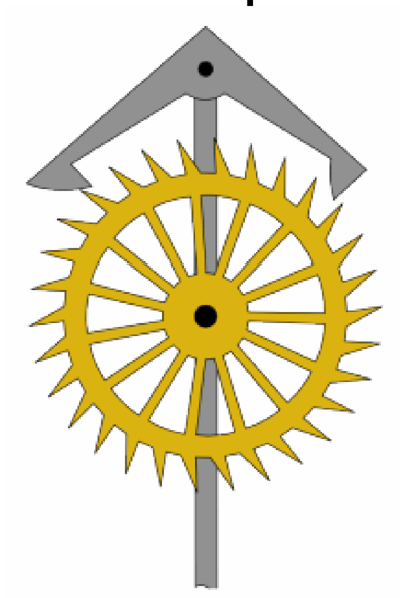
Verge **Anchor**	Mechanical clocks
Electronic oscillator circuit	electronic clocks
Microwave cavity attached to microwave oscillator, controlled by microprocessor	Atomic clocks

verge escapement



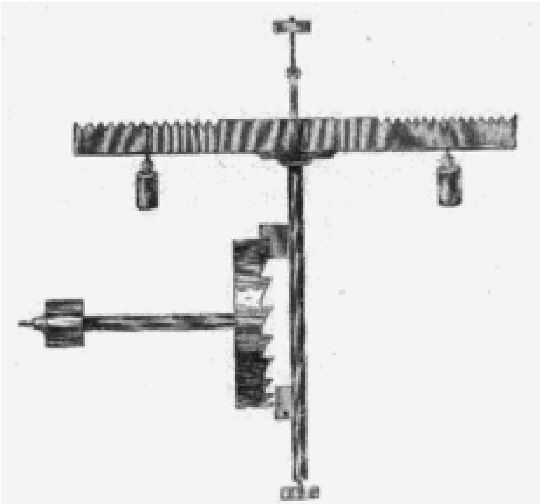


anchor escapement

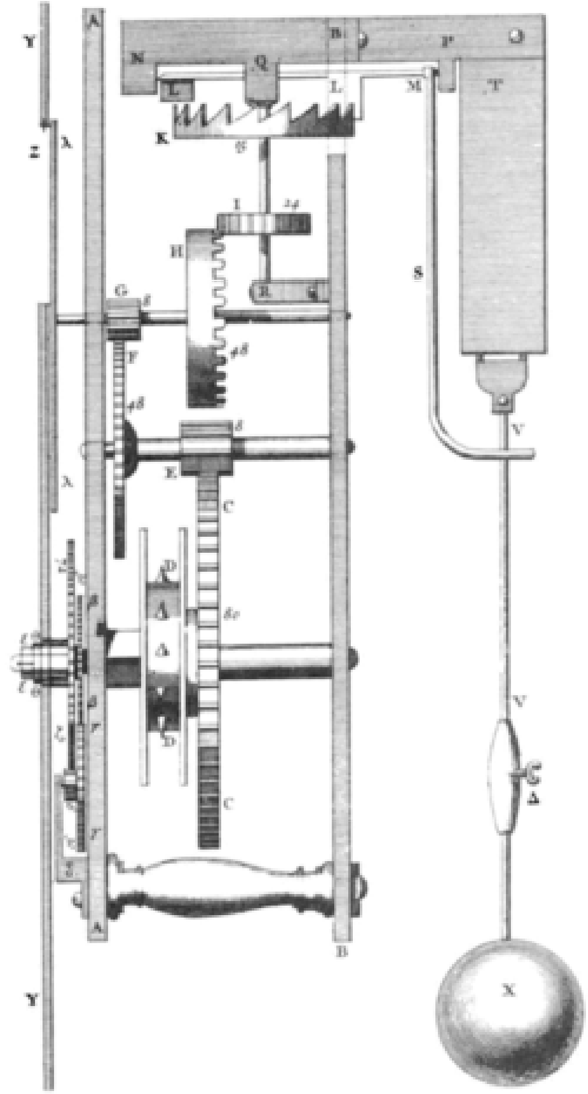


Clock verge escapement mechanisms

Verge with **balance wheel**



Huygens **cycloidal pendulum** clock (1673)



Verge with **spring**

THANK YOU

for your attention & patience